

Some Remarks on Lambert's Theorem. By H. C. Plummer, M.A.

1. It is a curious fact that the proofs of a theorem so celebrated as Lambert's, as they appear in at least the majority of text-books, are very unsatisfactory in the treatment given to the ambiguities of sign which must occur in the statement of the theorem. In most of the cases which arise in practice the arc of the orbit is comparatively short, and there is really no question about the signs to be chosen. This, coupled with the fact that it was commonly supposed that the form of the theorem was in general unsuitable for use in the determination of orbits, has apparently caused a certain carelessness in the demonstration. Recent work has, however, tended to place the value of the theorem more clearly in evidence.* For the case of elliptic motion the essence of the matter in regard to the distinction in the ambiguous signs has been given by Callandreau.† For hyperbolic motion under attraction to a centre a simple proof was given by Adams‡ in terms of hyperbolic functions, but the question of signs was ignored by him altogether. The case of motion under a repulsive force along the convex branch of an hyperbola does not seem to have been discussed at all, though the motion of matter in a comet's tail may give this case a real practical importance. It may therefore be useful to give here some examination to the methods of proof in the several cases, with such remarks as may be suggested incidentally.

2. The case of elliptic motion will be discussed first. Let E_1, E_2 be the eccentric anomalies of the starting and end points respectively, and let $E_2 > E_1$, both angles being less than 360° . In the first instance we suppose that the motion is in the same sense as the anomaly increases. It is easy to prove that, μ being the force at unit distance,

$$\mu^{\frac{1}{2}} a^{-\frac{3}{2}} t = \epsilon - \delta - (\sin \epsilon - \sin \delta)$$

where

$$\sin^2 \frac{1}{2} \epsilon = (r_1 + r_2 + c)/4a \equiv m_1^2, \quad \sin^2 \frac{1}{2} \delta = (r_1 + r_2 - c)/4a \equiv m_2^2$$

The relations of ϵ and δ to the eccentric anomalies are given by

$$\epsilon - \delta = E_2 - E_1, \quad \cos \frac{1}{2}(\epsilon + \delta) = e \cos \frac{1}{2}(E_2 + E_1)$$

The angle $\frac{1}{2}(\epsilon + \delta)$ is merely defined by the latter relation, and may be taken between the limits 0° and 180° ; and $\frac{1}{2}(\epsilon - \delta)$ must lie between the same limits. Hence ϵ must lie between 0° and 360° , while δ must lie between $\pm 180^\circ$. It follows that $\sin \frac{1}{2}\epsilon$ and $\cos \frac{1}{2}\delta$ are both positive. If, then, ϵ_1 and δ_1 are the principal values resulting from

$$\sin \frac{1}{2}\epsilon = +m_1, \quad \sin \frac{1}{2}\delta = +m_2$$

* Cf. *M.N.*, lxiii. p. 147.

† *Annales de l'Obs. de Paris, Mémoires*, vol. xxiii.

‡ *Collected Scientific Papers*, vol. i. p. 411.

we have to choose between ϵ_1 and $2\pi - \epsilon_1$ and between $\pm \delta_1$. Now the intercept x on the major axis between the chord joining the end points and the centre is given by

$$x = a \sec \frac{1}{2}(E_2 + E_1) \cos \frac{1}{2}(E_2 - E_1)$$

or

$$\begin{aligned} (x + ae)/(x - a) &= \left\{ \cos \frac{1}{2}(E_2 - E_1) + e \cos \frac{1}{2}(E_2 + E_1) \right\} \\ &\quad \div \left\{ \cos \frac{1}{2}(E_2 - E_1) - \cos \frac{1}{2}(E_2 + E_1) \right\} \\ &= \cos \frac{1}{2}\epsilon \cos \frac{1}{2}\delta / \sin \frac{1}{2}E_1 \sin \frac{1}{2}E_2 \end{aligned}$$

Under the conditions imposed, the sign of $\cos \frac{1}{2}\epsilon$ is the same as that of the fraction on the left: this shows that $\epsilon = \epsilon_1$ or $2\pi - \epsilon_1$ according as the chord intersects the axis outside or inside the range between the second focus and "perihelion." Similarly

$$(x - ae)/(x - a) = \sin \frac{1}{2}\epsilon \sin \frac{1}{2}\delta / \sin \frac{1}{2}E_2 \sin \frac{1}{2}E_1$$

which shows that $\delta = \delta_1$ or $-\delta_1$ according as the chord does not, or does, intersect the axis between the attracting focus and "perihelion."

3. Let now

$$\begin{aligned} \epsilon_1 - \delta_1 - (\sin \epsilon_1 - \sin \delta_1) &= \mu^{\frac{1}{2}} a^{-\frac{3}{2}} T_1 = nT_1 \\ \epsilon_1 + \delta_1 - (\sin \epsilon_1 + \sin \delta_1) &= \mu^{\frac{1}{2}} a^{-\frac{3}{2}} T_2 = nT_2 \end{aligned}$$

Then, under the conditions assumed, we have found that

$$t = T_1, 2\pi/n - T_2 \text{ or } 2\pi/n - T_1$$

according as the segment of which the curved boundary is described in the motion contains (1) no focus, (2) the empty but not the attracting focus, (3) both foci. But the conditions have precluded motion past "perihelion," and this condition must be removed. It is merely necessary to consider the motion over the complementary parts of the ellipse corresponding to the cases already distinguished. We thus find that

$$t = 2\pi/n - T_1, T_2 \text{ or } T_1$$

according as the segment contains (1) both foci, (2) the attracting but not the empty focus, (3) no focus. And now, by combining the six cases considered separately, we arrive at the final result for motion within a single circuit:

- (1) $t = T_1$ when the segment of which the curved boundary is described contains neither focus.
- (2) $t = 2\pi/n - T_1$ when the segment contains both foci.
- (3) $t = T_2$ when the segment contains the attracting but not the other focus.
- (4) $t = 2\pi/n - T_2$ when the segment contains the second but not the attracting focus.

4. If we are given merely the position of the attracting focus and of the two points 1, 2 on the orbit and the value of α , it is

easy to see that all the four cases enumerated are possible solutions, and that it is not possible to discriminate between them without further knowledge of the circumstances. For the second focus lies on the circle whose centre is 1 and radius $2a - r_1$; it also lies on the circle whose centre is 2 and radius $2a - r_2$. The necessary and sufficient condition that these circles shall intersect is clearly

$$\begin{aligned} 2a - r_1 + 2a - r_2 &> c \\ \text{i.e. } 4a &> r_1 + r_2 + c \end{aligned}$$

c being the length of the chord joining 1, 2.

If the condition (which, we may notice, entails the reality of ϵ and δ) be fulfilled, the ellipse will be real, and there are two positions which the second focus may occupy, one on either side of the chord joining 1, 2. Hence in the one case the foci are on the same side, in the other case on opposite sides of the chord. In the former we have case (1) or (2) according to the arc chosen in passing between 1 and 2; in the latter we have case (3) or (4) according as the motion takes place *via* "perihelion," or *via* "aphelion." The four cases are thus all possible, and the choice among them must be guided by additional knowledge. The four solutions are given by Gauss,* but he did not discuss the geometrical criteria which serve to distinguish them. Curiously enough, Callandreau himself omitted case (3), while the solutions given in the text-books are (1) and (3). When $4a = r_1 + r_2 + c$ the circles referred to above touch one another at a point between 1 and 2. There is then only one position of the second focus and the chord 12 passes through it. The four solutions become equal in pairs ($\epsilon = \pi$), as they also do when the attracting focus lies on the chord ($r_1 + r_2 = c$; $\delta = 0$).

5. Some difficulty arises in the calculation of t when ϵ and δ are small or nearly equal. The former case occurs when a is large, *i.e.* when the ellipse approaches parabolic eccentricity; the latter occurs when the chord c is small. Both these circumstances may concur. For elliptic motion I have previously† given a modification of the formula which avoids any uncertainty in the computation, and obviates resort to any special tables.

6. The formulæ for parabolic motion can be deduced from those already obtained for the ellipse. As the second focus approaches an infinite distance it cannot lie within the finite segment whose boundary is described in the motion. Hence we are limited to the cases (1) and (3) of § 3; ϵ has its principal value and δ is positive or negative according as the angular motion about the centre of attraction is less or greater than 180° . Hence in the limit

$$\begin{aligned} a\epsilon^2 &= r_1 + r_2 + c, & a\delta^2 &= r_1 + r_2 - c \\ 6\mu^{\frac{1}{2}}t &= a^{\frac{3}{2}}(\epsilon^3 - \delta^3) \\ &= (r_1 + r_2 + c)^{\frac{3}{2}} \mp (r_1 + r_2 - c)^{\frac{3}{2}} \end{aligned}$$

* *Theoria Motus*, § 107.

† *M.N.*, lxvi. p. 496.

the upper sign being taken when the segment does not contain the focus, the lower when it does contain the focus. This is Euler's equation, the use of which is much facilitated by Encke's transformation and auxiliary tables. This transformation is itself not free from ambiguities, and the question has been recently discussed by S. Tscherny.*

7. We consider next the case of motion along the concave branch of an hyperbola. Any point on one branch of an hyperbola referred to its centre as origin can be represented by

$$x = a \cosh u, \quad y = a(e^2 - 1)^{\frac{1}{2}} \sinh u$$

The focus for which the branch is concave is $(ae, 0)$. Hence

$$r^2 = a^2 \{ (e - \cosh u)^2 + (e^2 - 1)(\cosh^2 u - 1) \}$$

$$r = a(e \cosh u - 1)$$

and

$$c^2 = a^2 (\cosh u_2 - \cosh u_1)^2 + a^2 (e^2 - 1) (\sinh u_2 - \sinh u_1)^2$$

$$= 4a^2 \sinh^2 \frac{1}{2}(u_2 - u_1) \{ -1 + e^2 \cosh^2 \frac{1}{2}(u_2 + u_1) \}$$

$$= 4a^2 \sinh^2 \alpha \sinh^2 \beta$$

where

$$\alpha = \frac{1}{2}(u_2 - u_1), \quad \cosh \beta = e \cosh \frac{1}{2}(u_2 + u_1)$$

We may suppose $u_2 > u_1$ and therefore α positive; also of the two possible values of β , the positive can be chosen. Again

$$r_2 + r_1 = 2a(e \cosh \alpha \cosh \beta - 1)$$

If then

$$\epsilon = \beta + \alpha, \quad \delta = \beta - \alpha$$

$$(r_1 + r_2 + c)/4a \equiv m_1^2 = \sinh^2 \frac{1}{2}\epsilon, \quad (r_1 + r_2 - c)/4a \equiv m_2^2 = \sinh^2 \frac{1}{2}\delta$$

Now

$$ht = \int r^2 d\theta = \int (x dy - y dx)$$

$$= a^2 (e^2 - 1)^{\frac{1}{2}} \int \{ (\cosh u - e) \cosh u - \sinh^2 u \} du$$

$$= a^2 (e^2 - 1)^{\frac{1}{2}} \{ (u_1 - u_2) - e(\sinh u_1 - \sinh u_2) \}$$

$$= a^2 (e^2 - 1)^{\frac{1}{2}} (-2\alpha + 2 \sinh \alpha \cosh \beta)$$

$$= a^2 (e^2 - 1)^{\frac{1}{2}} \{ -(\epsilon - \delta) + (\sinh \epsilon - \sinh \delta) \}$$

where

$$h^2 = \mu a (e^2 - 1)$$

μ being the attracting force at unit distance. Hence

$$\mu^{\frac{1}{2}} a^{-\frac{3}{2}} t = \sinh \epsilon - \sinh \delta - (\epsilon - \delta)$$

where ϵ and δ are given by

$$\sinh \frac{1}{2}\epsilon = \pm m_1, \quad \sinh \frac{1}{2}\delta = \pm m_2$$

or

$$\exp. (\frac{1}{2}\epsilon) = \pm m_1 + (m_1^2 + 1)^{\frac{1}{2}}, \quad \exp. (\frac{1}{2}\delta) = \pm m_2 + (m_2^2 + 1)^{\frac{1}{2}}$$

* *A.N.*, No. 4280.

and it is necessary to decide between these pairs of positive, reciprocal values. Now ϵ is positive because a and β are both positive. Hence $\exp. (\frac{1}{2} \epsilon) > 1$ and $+m_1$ must be chosen. Again, the equation of the chord gives for the intercept on the axis beyond the focus

$$\begin{aligned} x - ae &= a \operatorname{sech} \frac{1}{2}(u_2 + u_1) \{ \cosh \frac{1}{2}(u_2 - u_1) - e \cosh \frac{1}{2}(u_2 + u_1) \} \\ &= -2a \operatorname{sech} \frac{1}{2}(u_2 + u_1) \sinh \frac{1}{2} \epsilon \sinh \frac{1}{2} \delta \end{aligned}$$

Hence, according as the chord intersects the axis beyond the focus or not, *i.e.* according as the angular motion is greater or less than 180° , δ is negative or positive, that is, $-m_2$ or $+m_2$ must be chosen. Accordingly

$$\begin{aligned} \mu^{\frac{1}{2}} a^{-\frac{3}{2}} t &= +2m_1(m_1^2 + 1)^{\frac{1}{2}} \mp 2m_2(m_2^2 + 1)^{\frac{1}{2}} \\ &\quad - 2 \operatorname{Log} \{ (m_1^2 + 1)^{\frac{1}{2}} + m_1 \} \pm 2 \operatorname{Log} \{ (m_2^2 + 1)^{\frac{1}{2}} + m_2 \} \end{aligned}$$

where Log denotes natural logarithm, and the upper or lower sign is to be taken according as the angular motion about the attracting focus is less or greater than 180° .

8. With any given (positive) value of a it is always possible to draw two hyperbolas such that the points 1 and 2 lie on the branch which is concave to the attracting centre. For the position of the second focus is determined by the intersections of two circles whose centres are at 1 and 2, and whose radii are $2a + r_1$ and $2a + r_2$ respectively. The intersections are always real, for the necessary condition

$$2a + r_1 + 2a + r_2 > c$$

is always fulfilled. They lie on opposite sides of the chord r_2 , and hence one focus must be on the same side as the attracting focus, the other on the opposite side of the chord. These facts correspond with the ambiguity of sign in the solution, which can only be removed, therefore, by independent knowledge of the geometrical circumstances of the motion.

9. To obtain a form more suitable for calculation, we put

$$m_1^2 = \sinh^2 \frac{1}{2} \epsilon = \tan^2 \mu_1, \quad m_2^2 = \sinh^2 \frac{1}{2} \delta = \tan^2 \mu_2$$

where μ_1 and μ_2 are to be taken in the first quadrant. Then we have

$$\begin{aligned} \frac{1}{2} \mu^{\frac{1}{2}} a^{-\frac{3}{2}} t &= \tan \mu_1 \sec \mu_1 \mp \tan \mu_2 \sec \mu_2 \\ &\quad - \operatorname{Log} \tan (45^\circ + \frac{1}{2} \mu_1) \pm \operatorname{Log} \tan (45^\circ + \frac{1}{2} \mu_2) \end{aligned}$$

In what follows only the upper signs will be retained. We can write

$$\begin{aligned} \mu^{\frac{1}{2}} a^{-\frac{3}{2}} t &= \sinh \epsilon - \sinh \delta - 2 \sinh \frac{1}{2}(\epsilon - \delta) \\ &\quad + 2 \{ \sinh \frac{1}{2}(\epsilon - \delta) - \frac{1}{2}(\epsilon - \delta) \} \\ &= 4 \sinh \frac{1}{2}(\epsilon - \delta) \sinh^2 \frac{1}{4}(\epsilon + \delta) + \frac{1}{2} \sinh^3 \frac{1}{2}(\epsilon - \delta). A \end{aligned}$$

where

$$A = 6 \{ \sinh \frac{1}{2}(\epsilon - \delta) - \frac{1}{2}(\epsilon - \delta) \} / \sinh^3 \frac{1}{2}(\epsilon - \delta)$$

and evidently tends to unity as $(\epsilon - \delta)$ becomes smaller. Now

$$c/2a = m_1^2 - m_2^2 = \sinh \frac{1}{2}(\epsilon - \delta) \sinh \frac{1}{2}(\epsilon + \delta)$$

and it can be found that, as a sufficient approximation when $(\epsilon - \delta)$ is small,

$$\log A = 3.6 \log \operatorname{sech} \frac{1}{4}(\epsilon - \delta)$$

Hence

$$\mu^{\frac{1}{2}}t = a^{\frac{1}{2}}c \tanh \frac{1}{4}(\epsilon + \delta) + \frac{1}{2^{\frac{1}{2}}}a^{-\frac{3}{2}}c^3 \sinh^{-3} \frac{1}{2}(\epsilon + \delta) \{ \operatorname{sech} \frac{1}{4}(\epsilon - \delta) \}^{3.6}$$

and we easily find that

$$\begin{aligned} \tanh \frac{1}{4}(\epsilon + \delta) &= \sin \frac{1}{2}(\mu_1 + \mu_2) / \cos \frac{1}{2}(\mu_1 - \mu_2) \\ \sinh \frac{1}{2}(\epsilon + \delta) &= 2 \sin \frac{1}{2}(\mu_1 + \mu_2) \cos \frac{1}{2}(\mu_1 - \mu_2) \sec \mu_1 \sec \mu_2 \\ \operatorname{sech} \frac{1}{4}(\epsilon - \delta) &= (\cos \mu_1 \cos \mu_2)^{\frac{1}{2}} \sec \frac{1}{2}(\mu_1 + \mu_2) \end{aligned}$$

The equation has thus been reduced to a form which can present no difficulty to the computer when the angles μ_1 and μ_2 become small or nearly equal, as they do when the hyperbola is nearly parabolic or the chord is small.

10. It has been found useful to employ the approximation

$$\begin{aligned} A &\equiv 6 \{ \sinh \frac{1}{2}(\epsilon - \delta) - \frac{1}{2}(\epsilon - \delta) \} / \sinh^3 \frac{1}{2}(\epsilon - \delta) \\ &= \{ \operatorname{sech} \frac{1}{4}(\epsilon - \delta) \}^{3.6} \end{aligned}$$

and it will be well to examine the nature of this approximation. We have

$$\begin{aligned} \operatorname{Log} (\sinh x - x) &= \operatorname{Log} (-i \sin ix - x) \\ &= \operatorname{Log} \left(\frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) \\ &= 3 \operatorname{Log} x - \operatorname{Log} 6 + S_1 \end{aligned}$$

where $i^2 = -1$ and *

$$S_1 = \frac{x^2}{20} - \frac{x^4}{16800} - \frac{x^6}{756000} + \dots$$

But

$$\operatorname{Log} \sinh x = \operatorname{Log} x + \frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835} - \dots$$

Hence writing x for $\frac{1}{2}(\epsilon - \delta)$ we find

$$\operatorname{Log} A = -\frac{9}{20}x^2 + \frac{93}{5600}x^4 - \frac{89}{84000}x^6 \dots$$

* Cf. *A.N.*, No. 3883, p. 300

Now

$$\text{Log cosh } x = \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{45} - \dots$$

so that

$$3.6 \text{ Log cosh } \frac{1}{2}x = \frac{9}{20}x^2 - \frac{3}{160}x^4 + \frac{1}{800}x^6 \dots$$

which shows that

$$\text{Log } A - 3.6 \text{ Log sech } \frac{1}{2}x = -\frac{3}{1400}x^4 + \frac{1}{5250}x^6 \dots$$

The residual terms only begin to be significant even in 7-figure logarithms when x is about 0.1, *i.e.* when $\epsilon - \delta$ is 0.2; and when the difference between ϵ and δ is so great there is no difficulty in the direct calculation of t . And it must be noted that the term of which A is a factor is itself small in comparison with the other term, and therefore A is not required with very great accuracy.

11. The method of § 7 can now be applied to the case of motion on the convex branch. The focus for which the same branch of the hyperbola is convex is at $(-ae, 0)$. In this case

$$r = a(e \cosh u + 1)$$

and, as before,

$$c = 2a \sinh \alpha \sinh \beta$$

where

$$\alpha = \frac{1}{2}(u_2 - u_1), \cosh \beta = e \cosh \frac{1}{2}(u_2 + u_1)$$

and α and β are both positive. If then

$$\epsilon = \beta + \alpha, \delta = \beta - \alpha$$

since

$$r_2 + r_1 = 2a(\cosh \alpha \cosh \beta + 1)$$

it follows that

$$m_1^2 \equiv (r_1 + r_2 + c)/4a = \cosh^2 \frac{1}{2}\epsilon, m_2^2 \equiv (r_1 + r_2 - c)/4a = \cosh^2 \frac{1}{2}\delta$$

Now

$$\begin{aligned} ht &= a^2(e^2 - 1)^{\frac{1}{2}} \int \{(\cosh u + e) \cosh u - \sinh^2 u\} du \\ &= a^2(e^2 - 1)^{\frac{1}{2}} \{ (u_2 - u_1) + e(\sinh u_2 - \sinh u_1) \} \\ &= a^2(e^2 - 1)^{\frac{1}{2}} \{ \epsilon - \delta + \sinh \epsilon - \sinh \delta \} \end{aligned}$$

where

$$h^2 = \mu a(e^2 - 1)$$

μ being the repulsive force at unit distance. Hence

$$\mu^{\frac{1}{2}} a^{-\frac{1}{2}} t = \epsilon - \delta + \sinh \epsilon - \sinh \delta$$

where ϵ and δ are given by

$$\cosh \frac{1}{2}\epsilon = +m_1, \cosh \frac{1}{2}\delta = +m_2$$

These are necessarily positive quantities, but two solutions are again possible in each case for

$$\exp. (\tfrac{1}{2}\epsilon) = m_1 \pm (m_1^2 - 1)^{\frac{1}{2}}, \exp. (\tfrac{1}{2}\delta) = m_2 \pm (m_2^2 - 1)^{\frac{1}{2}}$$

these being pairs of positive, reciprocal values. As before, ϵ is positive, and therefore $\exp. (\tfrac{1}{2}\epsilon) > 1$, and the upper sign must be taken in the first ambiguity. Also the intercept on the axis beyond the focus for which the branch is concave, in this case the empty focus, is as before

$$x - ae = -2a \operatorname{sech} \tfrac{1}{2}(u_2 + u_1) \sinh \tfrac{1}{2}\epsilon \sinh \tfrac{1}{2}\delta$$

which shows that δ is positive, *i.e.* $\exp. (\tfrac{1}{2}\delta) > 1$, if $x - ae$ is negative, that is, if the angular motion about the empty focus is less than 180° . Hence

$$\begin{aligned} \mu^{\frac{1}{2}}a^{-\frac{3}{2}}t = & 2 \operatorname{Log} \{m_1^2 - 1\}^{\frac{1}{2}} + m_1 \mp 2 \operatorname{Log} \{m_2^2 - 1\}^{\frac{1}{2}} + m_2 \\ & + 2m_1(m_1^2 - 1)^{\frac{1}{2}} \mp 2m_2(m_2^2 - 1)^{\frac{1}{2}} \end{aligned}$$

where the upper or lower sign is to be taken according as the angular motion about the empty focus is less or greater than 180° .

12. The second focus of the hyperbola in this case is one of the intersections of the circles whose centres are at 1 and 2, and whose radii are $r_1 - 2a$ and $r_2 - 2a$ respectively. In order that these circles may intersect it is necessary that

$$\begin{aligned} r_1 - 2a + r_2 - 2a &> c \\ \text{i.e.} \quad 4a &< r_1 + r_2 - c \end{aligned}$$

If this condition, which entails the reality of ϵ and δ , be fulfilled, two hyperbolas can be drawn so that the two points lie on the branch convex to the focus of repulsion. As in other cases, the two positions which the second focus may occupy are on opposite sides of the chord, and therefore in the one case it is on the same side as the first focus, in the other case on the opposite side. The ambiguity in the solution is inherent in the data.

13. To reduce the last equation to a form more suitable for calculation we put

$$m_1^2 - 1 = \tan^2 \mu_1, m_2^2 - 1 = \tan^2 \mu_2$$

where μ_1 and μ_2 are to be taken in the first quadrant. Then we have

$$\begin{aligned} \tfrac{1}{2}\mu^{\frac{1}{2}}a^{-\frac{3}{2}}t = & \tan \mu_1 \sec \mu_1 \mp \tan \mu_2 \sec \mu_2 \\ & + \operatorname{Log} \tan (45^\circ + \tfrac{1}{2}\mu_1) \mp \operatorname{Log} \tan (45^\circ + \tfrac{1}{2}\mu_2) \end{aligned}$$

The lower signs give a sum of four positive terms which cannot present much difficulty to calculate, and the case is not of practical importance. Retaining only the upper signs we find since

$$c/2a = m_1^2 - m_2^2 = \sin (\mu_1 - \mu_2) \sin (\mu_1 + \mu_2) \sec^2 \mu_1 \sec^2 \mu_2$$

that

$$\tan \mu_1 \sec \mu_1 - \tan \mu_2 \sec \mu_2 = c/4a \cdot (1 + \sin \mu_1 \sin \mu_2) \\ \div \sin \frac{1}{2}(\mu_1 + \mu_2) \cos \frac{1}{2}(\mu_1 - \mu_2)$$

and

$$\begin{aligned} & \text{Log } \tan (45^\circ + \frac{1}{2}\mu_1) - \text{Log } \tan (45^\circ + \frac{1}{2}\mu_2) \\ &= \text{Log} \{ \cos \frac{1}{2}(\mu_1 + \mu_2) + \sin \frac{1}{2}(\mu_1 - \mu_2) \} - \text{Log} \{ \cos \frac{1}{2}(\mu_1 + \mu_2) \\ & \quad - \sin \frac{1}{2}(\mu_1 - \mu_2) \} \\ &= \text{Log } (1 + x) - \text{Log } (1 - x) \end{aligned}$$

where

$$\begin{aligned} x &= \sin (\mu_1 - \mu_2)/2 \cos \frac{1}{2}(\mu_1 + \mu_2) \cos \frac{1}{2}(\mu_1 - \mu_2) \\ &= c/4a \cdot \cos^2 \mu_1 \cos^2 \mu_2 / \sin (\mu_1 + \mu_2) \cos \frac{1}{2}(\mu_1 + \mu_2) \cos \frac{1}{2}(\mu_1 - \mu_2) \end{aligned}$$

Thus x can be calculated accurately however small μ_1 and μ_2 may be. Both parts of the expression for t have therefore been reduced to a form free from difficulty.

14. If the value of x is very small, as it may be for a short arc, an approximation to the value of $\text{Log } (1 + x) - \text{Log } (1 - x)$ can be found thus. We have

$$\text{Log } (1 + x) - \text{Log } (1 - x) = 2 \left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \dots \right)$$

But

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots$$

Hence

$$\text{Log } (1 + x) - \text{Log } (1 - x) - 2 \tan x = \frac{2}{15}x^5 + \frac{8}{45}x^7 + \dots$$

The approximation $2 \tan (x. 206265'')$ holds therefore with fair accuracy if $x < 0.1$; and from 0.1 onwards the function can be calculated directly under the form

$$\text{Log } (1 + x) - \text{Log } (1 - x) = \text{Log } \tan (45^\circ + \tan^{-1} x)$$

Note that throughout Log has been used to denote natural logarithm, and log to denote common logarithm, so that

$$\text{Log} = 2.3025851 \log = [0.3622157] \log$$

University Observatory, Oxford:
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Note added 1909 January 19.

The approximation given in § 14 is of some importance, and may be useful for other purposes, *e.g.* in calculating the Guder-

mannian function for small values of the argument. It may be well, therefore, to examine it a little more closely. We have

$$\begin{aligned}\text{Log}\{\text{Log}(1+x) - \text{Log}(1-x)\} &= \text{Log} 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right) \\ &= \text{Log } 2x + R\end{aligned}$$

where

$$\begin{aligned}R &= \text{Log}\left(1 + \frac{x^2}{3} + \frac{x^4}{5} + \dots\right) \\ &= \frac{1}{3}x^2 + \frac{13}{90}x^4 + \frac{251}{2835}x^6 + \frac{3551}{56700}x^8 + \dots\end{aligned}$$

On the other hand,

$$\text{Log } \tan x = \text{Log } x + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 + \dots$$

Hence

$$\begin{aligned}\text{Log}\{\text{Log}(1+x) - \text{Log}(1-x)\} - \text{Log}(2 \tan x) \\ &= \frac{1}{15}x^4 + \frac{1}{15}x^6 + \frac{317}{5670}x^8 + \dots \\ &= x^4/15(1-x^2), \text{ very nearly.}\end{aligned}$$

Changing to common logarithms we have then

$$\begin{aligned}\log\{\text{Log}(1+x) - \text{Log}(1-x)\} &= \log(2 \tan x) \\ &\quad + [8.4616930]x^4/(1-x^2)\end{aligned}$$

and the following little table gives the correction additive to $\log(2 \tan x)$ in the seventh decimal place:—

<i>x.</i>	Corr.	<i>x.</i>	Corr.	<i>x.</i>	Corr.
0.04	001	0.12	061	0.16	195
.05	002	.13	084	.165	221
.06	004	.135	098	.17	249
.07	007	.14	113	.175	280
.08	012	.145	131	.18	314
.09	019	.15	150	.185	351
.10	029	.155	171	.19	391
.11	043	.16	195	.195	435
.12	061			.20	483

Thus, for example, when $x=0.2$, which is the circular measure of $11^\circ 27' 32''.96$, we have $\log(\text{Log } 1.2/0.8)=9.6079536$ and $\log(2 \tan x)=9.6079053$, the difference coinciding exactly with the last entry in the table.

A Method for the Reduction of Comet Photographs.

By H. C. Plummer, M.A.

1. The magnificent results which have lately been obtained in comet photography suggest some reflections on the method by which the plates may be reduced. The method of Bessel,* which was designed for the treatment of Halley's comet in 1835, and has been followed more or less faithfully by all subsequent writers, deals with observed position-angles and distances with respect to the nucleus. If the observations are not made in this form, they must first be reduced to it. Of course position-angles and distances can be very easily deduced from the photographic plate.† But it has frequently been found‡ that there is an advantage in the use of rectangular coordinates over spherical geometry in dealing with photographs. There is a corresponding change in the method of calculation from the general use of logarithms to the calculating machine which appeals strongly to some computers. It may, therefore, be worth while to investigate the formulæ which seem to be the natural expression of this point of view. Whether the emanations from a comet remain strictly in the plane of the orbit is open to doubt, but provisionally it is necessary to make this assumption, and it forms the basis of what follows.

2. It will be convenient to give precepts for certain preliminary operations. Let us form the arrays

$$\begin{vmatrix} -\sin A & \cos A & 0 \\ -\sin D \cos A & -\sin D \sin A & \cos D \\ \cos D \cos A & \cos D \sin A & \sin D \end{vmatrix} \quad . \quad (1)$$

where A, D are the R.A. and declination of the plate-centre, and

$$\begin{vmatrix} X & Y & Z \\ a \sin(\alpha + u) & b \sin(\beta + u) & c \sin(\gamma + u) \\ a \cos(\alpha + u) & b \cos(\beta + u) & c \cos(\gamma + u) \end{vmatrix} \quad . \quad (2)$$

where X, Y, Z are the rectangular coordinates of the Sun referred to the equatorial system of coordinates, $a, b, c, \alpha, \beta, \gamma$ the Gaussian constants for the orbit of the comet, and u an angle measured from the ascending node in the direction of the comet's motion: the actual value to be assigned to u can be left for definition later on. Next we form the array

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (3)$$

* *A.N.*, No. 300.† Cf. *M.N.*, lxi. p. 100.‡ Cf. *M.N.*, lx. pp. 176 and 201; lxii. p. 29.